Light pseudo-Goldstone bosons without explicit symmetry breaking

Antonio O. Bouzas *
Departamento de Física Aplicada, CINVESTAV-IPN
Carretera Antigua a Progreso Km. 6, Apdo. Postal 73 "Cordemex"
Mérida 97310, Yucatán, México

September, 2006

Abstract

A mechanism is discussed to obtain light scalar fields from a spontaneously broken continuous symmetry without explicitly breaking it. If there is a continuous manifold of classical vacua in orbit space, its tangent directions describe classically massless fields that may acquire mass from perturbations of the potential that do not break the symmetry. We consider the simplest possible example, involving a scalar field in the adjoint representation of SU(N). We study the scalar mass spectrum and its RG running at one-loop level including scalar and pseudoscalar Yukawa couplings to a massive Dirac fermion.

1 Introduction

In a theory with a global or local symmetry group G spontaneously broken by a multiplet of scalar fields, the classical vacuum manifold $\mathcal{Y}$ is globally invariant under the action of G. The set $\mathcal{Y}/\mathrm{G}$ of classical minima in orbit space is often discrete. When $\mathcal{Y}/\mathrm{G}$ is continuous its tangent directions are null vectors of the second derivative of the potential and, therefore, describe classically massless modes which are not Goldstone bosons (henceforth referred to as GBs) associated to the spontaneous breaking of G. It may happen that these tangent modes are GBs of a larger symmetry group $\mathrm{G}' \supset \mathrm{G}$ of the scalar sector, explicitly broken by other interaction terms invariant under G but not G'. In other words, they may be pseudo-Goldstone bosons (henceforth PGBs) related to G'. If no such larger symmetry G' exists, the tangent modes may acquire mass from additional small G-invariant terms in the scalar potential which cause deformations of $\mathcal{Y}$. We have then light scalar modes without explicitly breaking the symmetries of the theory, neither at tree level nor radiatively.

In this paper we consider the simplest possible model of light PGBs without explicit symmetry breaking, involving a scalar field in the adjoint representation of $\mathrm{SU}(N)$, $N > 3$. We study the mass spectrum of the model, and its renormalization when the scalar fields couple to Dirac fermions. In dimensional regularization fermion loops may have a large effect on the renormalization group (henceforth RG) running of scalar couplings and masses. In particular, as discussed below, fermionic radiative corrections may cause the mass of tangent modes to be decreasing functions of the renormalization scale. Because our aim is to investigate the mechanism outlined above to obtain light scalar fields, for simplicity we omit gauge interactions.

In section 2 below we briefly review the extremum analysis for a quartic potential in the adjoint representation of $\mathrm{SU}(N)$. In section 3 we consider the classical mass spectrum in the vacua of interest to us. The renormalization group evolution of that spectrum is studied in section 4, where we also consider in detail the conditions for stability of the vacuum. We try to keep the overlap of these sections with the previous literature, particularly [1, 2], to a minimum. Final remarks are given in section 5, and some related material is gathered in two appendices.

*E-mail: abouzas@mda.cinvestav.mx

2 Classical vacua

We consider a scalar field $\phi(x)$ in the adjoint representation of SU(N), $N \geq 3$. The most general renormalizable potential $V_N : \mathrm{su}(N) \to \mathbb{R}$ invariant under SU(N) is,

$$V_N(\phi) = -\frac{m^2}{2}\mathrm{Tr}\left(\phi^2\right) + c\mathrm{Tr}\left(\phi^3\right) + a(\mathrm{Tr}\left(\phi^2\right))^2 + b\mathrm{Tr}\left(\phi^4\right) \ . \tag{1}$$

We restrict ourselves to the strictly renormalizable case, $a \neq 0 \neq b$, with spontaneous symmetry breaking, $m^2 > 0$. We introduce the reduced potential $V_N(x) \equiv V_N(\mathrm{diag}(x_1, \ldots, x_N))$, with $\sum_{j=1}^N x_j = 0$. Ignoring the tracelessness constraint for the moment, the condition for $V_N(x)$ to be bounded below is that $V_N^{(4)}(x) \equiv a(\sum_{j=1}^N x_j^2)^2 + b\sum_{j=1}^N x_j^4 > 0$ for all x. Since $V_N^{(4)}(x)$ is an homogeneous function of x of even degree, this is equivalent to requiring that the minimum value of $V_N^{(4)}(x)$ over the unit sphere be positive. That minimum value is $a+b$ if $a > 0$ and $Na + b$ if $a < 0$, so a necessary and sufficient condition for V_N to be bounded below is

$$a + b > 0 \quad \text{and} \quad Na + b > 0 \ . \tag{2}$$

When the constraint $\sum_{j=1}^N x_j = 0$ is taken into account the conditions (2) are only sufficient, not necessary. We will always assume, nevertheless, that a and b are chosen so that (2) are satisfied.

To find the extrema of $V_N(x)$ we consider the unconstrained extremization of $\mathcal{L}_N(x, \lambda) = V_N(x_1, \ldots, x_N) + \lambda \sum_{j=1}^N x_j = 0$, with λ a Lagrange multiplier. $\mathcal{L}_N(x, \lambda)$ is a quartic symmetric polynomial in x_i. If x^o is an extremum of $\mathcal{L}_N$, $N \geq 4$, there cannot be more than three different values among its components if $b \neq 0$. To see this, we assume otherwise. Since $\mathcal{L}_N$ is invariant under permutations of x_i, we can always assume that x^o is such that ${x^o}_1 < {x^o}_2 < {x^o}_3 < {x^o}_4$. Thus, x^o is a point interior to the open domain $\{x_1 < x_2 < x_3 < x_4\} \subset \mathbb{R}^N$, and within that domain we can use as coordinates $s_1, s_2, s_3, s_4, x_5, \ldots, x_N$, with $s_k = \sum_{j=1}^4 x_j^k$. A necessary condition for x^o to be an extremum is then $(\partial\mathcal{L}_N/\partial s_4)|_{x^o} = b = 0$. Therefore, for $b \neq 0$ all extrema of the potential must be either of the form,

$$x^o = (\underbrace{\eta_1, \ldots, \eta_1}_{n_1}, \underbrace{\eta_2, \ldots, \eta_2}_{n_2}, \underbrace{\eta_3, \ldots, \eta_3}_{n_3}) \ , \quad n_1 + n_2 + n_3 = N \ , \quad n_{1,2,3} \geq 1 \tag{3a}$$

$$n_1\eta_1 + n_2\eta_2 + n_3\eta_3 = 0 \ , \tag{3b}$$

or of the form

$$x^o = (\underbrace{\eta_1, \ldots, \eta_1}_{k_1}, \underbrace{\eta_2, \ldots, \eta_2}_{k_2}) \ , \quad k_1 + k_2 = N \ , \quad k_{1,2} \geq 1 \ , \quad k_1\eta_1 + k_2\eta_2 = 0 \ . \tag{3c}$$

Since V_N is symmetric in its arguments x_i, any permutation of the components of x^o is also an extremum. We can then adopt the convention $n_1 \geq n_2 \geq n_3$. In some exceptional cases discussed below, an extremum can be both of type (3a) (with, say, $\eta_1 = \eta_2$) and of type (3c).

In the case of extrema x^o of the form (3a) with $\eta_i \neq \eta_j$ if $i \neq j$ $(i, j = 1, 2, 3)$, we can argue as above to conclude that η_i must satisfy $(\partial\mathcal{L}_N/\partial s_3)|_{x^o} = (c + 4/3bs_1)|_{x^o} = 0$ or, more explicitly,

$$\eta_1 + \eta_2 + \eta_3 = -\frac{3}{4}\frac{c}{b} \ . \tag{4}$$

Taking into account (3a), (3b), and (4), the extremum equations $\partial\mathcal{L}_N/\partial x_i = 0$ can be reduced to

$$(2n_1a + b)\eta_1^2 + (2n_2a + b)\eta_2^2 + (2n_3a + b)\eta_3^2 - \frac{m^2}{2} - \frac{9c^2}{16b} = 0 \ . \tag{5}$$

The constrained extrema of $V_N(x)$ are then either of the form (3a), with $\eta_{1,2,3}$ determined by (3b), (4) and (5), or of the form (3c). The latter can be obtained by setting $n_3 = 0$ in the solutions of type (3a), with eqs. (3b) and (5) determining $\eta_{1,2}$ in terms of η_3 as defined by (4). In order to find the extrema of $V_N(x)$ we consider the case $c = 0$ first, and the general case afterwards. Since $V_N(x)$ is invariant under the transformation $x \to -x$, $c \to -c$, we can restrict ourselves to $c \geq 0$.

Even potential ($c = 0$) The solutions to eqs. (3b), (4), (5) in this case are, with notation analogous to (3a),

$$y^0{}_{n_1,n_2,n_3} = (\underbrace{\xi_1,\ldots,\xi_1}_{n_1},\underbrace{\xi_2,\ldots,\xi_2}_{n_2},\underbrace{\xi_3,\ldots,\xi_3}_{n_3}) , \quad n_1+n_2+n_3 = N , \quad n_{1,2,3} \geq 1 \tag{6a}$$

$$\xi_1 = \frac{m}{2}\frac{n_2-n_3}{\sqrt{\Omega_{n_1,n_2,n_3}}} , \quad \xi_2 = \frac{m}{2}\frac{n_3-n_1}{\sqrt{\Omega_{n_1,n_2,n_3}}} , \quad \xi_3 = \frac{m}{2}\frac{n_1-n_2}{\sqrt{\Omega_{n_1,n_2,n_3}}} , \tag{6b}$$

$$\begin{gathered}\Omega_{n_1,n_2,n_3} = \Omega_a^{n_1,n_2,n_3} a + \Omega_b^{n_1,n_2,n_3} b, \quad \Omega_a^{n_1,n_2,n_3} = N(n_1n_2+n_2n_3+n_3n_1) - 9n_1n_2n_3, \\ \Omega_b^{n_1,n_2,n_3} = N^2 - 3(n_1n_2+n_2n_3+n_3n_1) .\end{gathered} \tag{6c}$$

Besides $\pm y^0{}_{n_1,n_2,n_3}$, all permutations of their components are also solutions to the extremum eqs. Clearly, if $n_1 = n_2 = n_3$ (6) reduces to the trivial solution. If not all n_i are equal it is possible to prove that $0 < \Omega_a^{n_1,n_2,n_3}$, $0 < \Omega_b^{n_1,n_2,n_3}$, and $1 \leq \Omega_a^{n_1,n_2,n_3}/\Omega_b^{n_1,n_2,n_3} \leq N$, provided (2) holds. Therefore $\Omega_{n_1,n_2,n_3} > 0$, and these extrema exist.

If $N = 3n$ it is possible to have $n_1 = n_2 = n_3 = n$. In this case eqs. (3b) and (4) are not independent so, instead of a discrete set, the extrema lie on the curve $\mathcal{Y}^0{}_{n,n,n}$ defined by

$$\xi_1+\xi_2+\xi_3 = 0 , \qquad \xi_1^2+\xi_2^2+\xi_3^2 = \kappa_n^2 , \qquad \left(\kappa_n^2 \equiv \frac{m^2}{2(2na+b)}\right) . \tag{7}$$

If (2) holds, then $(2na+b) > 0$. For $N = 3$ these are minima.

We can find all two-valued extrema $y^0{}_{n_1,n_2}$ directly by setting $n_3 = 0$ in (6), $y^0{}_{n_1,n_2} \equiv y^0{}_{n_1,n_2,0}$. Using (6) it can be shown that there cannot be integers $n_{1,2,3} \geq 1$ and $k_{1,2} \geq 1$ with $n_1+n_2+n_3 = N = k_1+k_2$ such that $y^0{}_{n_1,n_2,n_3}$ and $y^0{}_{k_1,k_2}$ are equal, up to a permutation of their components. If $N = 3n$, however, some of the $\mathcal{Y}^0{}_{n,n,n}$ may have only two different values among their components. In fact, $y^0{}_{2n,n}$ with components $\xi_{1,2}$ as given by (6b) with $n_3 = 0$ are easily seen to satisfy (7), with $\xi_1' = \xi_1$, $\xi_2' = \xi_1$, $\xi_3' = \xi_2$. Thus, $\pm y^0{}_{2n,n}$ and their permutations are part of the set $\mathcal{Y}^0{}_{n,n,n}$.

Non-even potential ($c > 0$) The solutions to the non-homogeneous linear eqs. (3b) and (4) can be written as $\eta_i = \sigma\xi_i + \rho_i$, $i = 1,2,3$, with ξ_i the solution to the homogeneous eqs. given by (6), σ a free real parameter, and ρ_i a particular solution to (3b) and (4). In the case $n_1 > n_2 > n_3 \geq 1$, ρ_i can be taken to be,

$$\rho_1 = \frac{3c}{4b}\frac{n_2n_3}{n_{12}n_{31}} , \quad \rho_2 = \frac{3c}{4b}\frac{n_1n_3}{n_{12}n_{23}} , \quad \rho_3 = \frac{3c}{4b}\frac{n_1n_2}{n_{23}n_{31}} , \quad n_{ij} \equiv n_i - n_j , \tag{8}$$

and similarly in the cases $n_2 = n_3$ and $n_1 = n_2$. There are no solutions with $n_1 = n_2 = n_3$. The parameter σ is determined from (5), which reduces to

$$\frac{m^2}{4}\sigma^2 + \sum_{i=1}^{3}(2an_i+b)\xi_i\rho_i\sigma + \frac{1}{2}\sum_{j=1}^{3}(2an_j+b)\rho_j^2 = \frac{m^2}{4} + \frac{9c^2}{32b} . \tag{9}$$

The extrema of type (3a) are then explicitly determined by (6), (8), and the two solutions to (9). There are also extrema of the form (3c) wich are given by (6) and (8), with $n_3 = 0$, and

$$\sigma^2 - 2z\sigma - 1 = 0 , \qquad z = \frac{3c}{4m}\frac{n_1-n_2}{\sqrt{\Omega_{n_1,n_2,0}}} . \tag{10}$$

We will denote the two possible values of σ by $\sigma_\pm = z \pm \sqrt{z^2+1}$, and the corresponding extrema as $x^0{}_{n_1,n_2,\pm}$. Since in this case $\rho_{1,2} = 0$, we have $x^0{}_{n_1,n_2,\pm} = \sigma_\pm y^0{}_{n_1,n_2}$.

Provided the conditions (2) are satisfied, for fixed values of a, $b \neq 0$, $m^2 > 0$, and $n_{1,2,3} \geq 1$, there exists at least one value of c^2 such that two of the η_i in (3a) are equal. For those values of c the extremum x^0 can be written both in the form (3a) (with $n_3 \geq 1$) and (3c). We will not dwell longer on this issue because it will not be of importance in what follows and, furthermore, because such fine tuning of parameters is not preserved under renormalization.

3 Mass spectrum

At each extremum x^0 the Hessian matrix for the potential $V_N(x)$ can be diagonalized (see appendix A) to obtain the corresponding mass spectrum. In the case $c = 0$ this diagonalization can be carried out explicitly and the extrema ${y^0}_{n_1,n_2,n_3}$ and ${y^0}_{n_1,n_2}$ completely classified. When $c \neq 0$ a complete classification is more difficult to obtain by direct computation. A proof that all extrema ${x^0}_{n_1,n_2,n_3}$ with three different components are saddle points is given in [2]. At an extremum ${x^0}_{n_1,n_2,\pm}$ ($n_1 \geq n_2 \geq 1$) the Hessian matrix has the four eigenvalues,

$$
\begin{aligned}
&\omega_1 = 0 \ , && [1] && \{2n_1n_2 + 1\} \\
&\omega_2 = m^2(\sigma_\pm^2 + 1) \ , && [1] && \{1\} \\
&\omega_3 = m^2 N \left(\frac{2z\sigma_\pm}{n_1 - n_2} + \frac{2n_2 - n_1}{\Omega_{n_1,n_2,0}} \sigma_\pm^2 b \right) \ , && [n_1 - 1] && \{n_1^2 - 1\} \\
&\omega_4 = m^2 N \left(-\frac{2z\sigma_\pm}{n_1 - n_2} + \frac{2n_1 - n_2}{\Omega_{n_1,n_2,0}} \sigma_\pm^2 b \right) \ , && [n_2 - 1] && \{n_2^2 - 1\} \ .
\end{aligned}
\tag{11}
$$

The square brackets indicate "reduced" multiplicities, corresponding to the Hessian matrix of the reduced potential $V_N(x)$. These are multiplicities in orbit space (see appendix B). The "total" multiplicities obtained from the Hessian matrix of the full potential $V_N(\phi)$ on su(N), indicated in (11) in curly brackets, can be computed from reduced multiplicities (appendix B), or simply by noticing that the stability group of ${x^0}_{n_1,n_2,\pm}$ is SU(n_1) × SU(n_2) × U(1) if $n_2 > 1$, and SU(n_1) × U(1) if $n_2 = 1$. One null eigenvalue (ω_1 in (11)) is an artifact of the projection on the constrained subspace $\sum_{i=1}^N x_i = 0$ (see appendix A). The eigenvalues $\omega_{2,3,4}$ give the squared masses of three massive multiplets. The mass-squared average and difference between the SU(n_1) and SU(n_2) multiplets is[1]

$$
\frac{1}{2}(\omega_3 + \omega_4) = \frac{m^2}{2} \frac{N^2 b}{\Omega_{n_1,n_2,0}} \sigma_\pm^2 \ , \qquad \omega_3 - \omega_4 = m^2 N \left(\frac{4z\sigma_\pm}{n_1 - n_2} - 3\frac{n_1 - n_2}{\Omega_{n_1,n_2,0}} b\sigma_\pm^2 \right) \ . \tag{12}
$$

Notice that $\omega_3 + \omega_4 < 0$ if $b < 0$ and $n_2 > 1$, so ${x^0}_{n_1,n_2,\pm}$ cannot be minima in that case . In some open regions of the plane (a, b) it is possible to choose $c = c(a, b)$ so that $\omega_3 - \omega_4 = 0$ but, with the exception of the case $n_1 = n_2$, $c = 0$, such fine tuning is not preserved under renormalization.

For even potentials ($c = 0$) if $N = 3n$ there is an extremal manifold ${\mathcal{Y}^0}_{n,n,n}$ defined by (7), containing in particular the extrema $\pm {y^0}_{2n,n}$. We are interested in the mass spectrum at those extrema and, for $c \neq 0$, at the related extrema ${x^0}_{2n,n,\pm}$. We consider the case $N = 3n$ with $n > 1$ first. The mass-squared spectrum of $V_N(x)|_{c=0}$ at the extrema satisfying (7) and such that $\xi_i \neq \xi_j$ if $i \neq j$ is (see appendix A)

$$
\begin{gathered}
\widetilde{\omega}_1 = 0 \ , \quad [2]\{6n^2 + 2\} \ , \qquad \widetilde{\omega}_2 = 2m^2 \ , \quad [1]\{1\} \ , \\
\widetilde{\omega}_3 = 12b \left(\xi_1^2 - \frac{\kappa_n^2}{6} \right) \ , \quad \widetilde{\omega}_4 = 12b \left(\xi_2^2 - \frac{\kappa_n^2}{6} \right) \ , \quad \widetilde{\omega}_5 = 12b \left(\xi_3^2 - \frac{\kappa_n^2}{6} \right), \quad [n-1]\{n^2 - 1\} \ ,
\end{gathered}
\tag{13}
$$

where we used the same notation as in (11) and κ_n^2 is defined in (7). All the eigenvalues on the last line have the same multiplicity. The null eigenvalue $\widetilde{\omega}_1$ has two eigenvectors in orbit space, su(N)/SU(N), as indicated in (13). One of them is a spurious projection mode. The other one lies on the subspace tangent to the 1-dimensional manifold defined by (7), on which $V_N(x)|_{c=0}$ is constant. The stability group of these extrema is SU(n)^3 × U(1)^2, so we expect to have $6n^2$ GBs plus one spurious massless mode, three massive SU(n) multiplets, and two independent massive U(1) modes. Out of the latter, one lies in the tangent subspace and remains massless at tree level. Since it is impossible to satisfy (7) and to have $(\xi_i^2 - \kappa_n^2/6) > 0$ (or < 0) simultaneously for $i = 1, 2, 3$, these extrema are saddle points for any values of a and b.

The extrema of $V_N(x)|_{c=0}$ satisfying (7) with $\xi_1 = \xi_2$ are $\pm {y^0}_{2n,n}$ and their permutations, with stability group SU($2n$) × SU(n) × U(1). The eigenvalue spectrum at ${y^0}_{2n,n}$ can be obtained either from (11) (with $c = 0$, $n_1 = 2n_2 = 2n$) or from (13) (with $\xi_1 = \xi_2 = \pm\kappa_n/\sqrt{6}$, $\xi_3 = -2\xi_1$). The result is that there is a null eigenvalue with total multiplicity $8n^2$ comprising $4n^2$ GBs, one spurious projection mode, and $4n^2 - 1$ PGBs. The latter form an SU($2n$) multiplet containing the mode lying in the tangent direction to the manifold (7)

[1] The average mass in (12) is slightly different from the result in [2].

in orbit space. Since the tangent mode is massless at tree level, the mass of the entire multiplet must vanish. The spectrum contains also a U(1) mode of squared mass $2m^2$ and an SU(n) multiplet with n^2-1 modes of mass $6b\kappa_n^2$. We see that for $b>0$ all Hessian eigenvalues at $\pm {y^0}_{2n,n}$ are non-negative. Expanding $V_N(x)|_{c=0}$ in power series about $\pm {y^0}_{2n,n}$ through third order, however, shows that these extrema are not minima but saddle points.

When $c\neq 0$ there are no solutions to (7). The only extrema related to the extremal manifold ${\mathcal{Y}^0}_{n,n,n}$ are ${x^0}_{2n,n,\pm}$ as defined in §2, with ${x^0}_{2n,n,\pm} \xrightarrow{c\to 0} \pm {y^0}_{2n,n}$. As discussed above, we need only consider the case $c>0$. In this case it is immediate from (11) that ${x^0}_{2n,n,-}$ cannot be a minimum, because at it $\omega_3<0$. On the other hand, at ${x^0}_{2n,n,+}$ the non-zero eigenvalues in the spectrum (11) reduce to

$$\begin{gathered}\omega_2 = m^2(1+\sigma_+^2) = 2m^2 + \mathcal{O}\left(cm\right)\ , \qquad \omega_3 = 6m^2 z\sigma_+ = \frac{3\sqrt{3}}{2}\frac{cm}{\sqrt{2na+b}} + \mathcal{O}\left(c^2\right)\ , \\ \omega_4 = 6\kappa_n^2(2na(1-\sigma_+^2)+b) = 6\kappa_n^2 b + \mathcal{O}\left(cm\right)\ ,\end{gathered} \tag{14}$$

with κ_n^2 defined in (7) and z and σ_+ in (10). The counting of modes is the same as in the case of ${y^0}_{2n,n}$ discussed above, except that in (14) the SU($2n$) multiplet has a mass $\sqrt{\omega_3} = \mathcal{O}(\sqrt{cm})$. The eigenvalue ω_2 is obviously positive and, since we are assuming $c>0$ and (2), also $\omega_3>0$. The conditions on the coupling constants for $\omega_4>0$, and for $\omega_4 \lesseqgtr \omega_3$, can be summarized as follows

$$\begin{aligned}
&a>0\ , \quad b>0 \quad \text{and} \quad \begin{cases} 0<\frac{c}{m}<\sqrt{\frac{2}{3}}\frac{b}{\sqrt{4na+b}} & \Rightarrow \omega_4>\omega_3>0\ , \\ \sqrt{\frac{2}{3}}\frac{b}{\sqrt{4na+b}}<\frac{c}{m}<\sqrt{\frac{2}{3na}}b & \Rightarrow \omega_3>\omega_4>0\ , \\ \sqrt{\frac{2}{3na}}b<\frac{c}{m} & \Rightarrow \omega_3>0>\omega_4\ , \end{cases} \\
&a>0\ , \quad b<0 \Rightarrow \omega_3>0>\omega_4\ , \\
&a<0\ , \quad b>0 \quad \text{and} \quad \begin{cases} 4na+b<0 & \Rightarrow \omega_4>\omega_3>0\ , \\ 4na+b>0 \quad \text{and} \quad \frac{c}{m}<\sqrt{\frac{2}{3}}\frac{b}{\sqrt{4na+b}} & \Rightarrow \omega_4>\omega_3>0\ , \\ 4na+b>0 \quad \text{and} \quad \sqrt{\frac{2}{3}}\frac{b}{\sqrt{4na+b}}<\frac{c}{m} & \Rightarrow \omega_3>\omega_4>0\ . \end{cases}
\end{aligned} \tag{15}$$

We remark that (15) holds under the assumptions $N=3n$, $n>1$, $c>0$, and (2).

In the case $N=3$ ($n=1$), the eigenvalues of the Hessian matrix of $V_3(x)|_{c=0}$ at the extremal manifold ${\mathcal{Y}^0}_{1,1,1}$ defined by (7) are $\widetilde{\omega}_1=0$ (with reduced and total multiplicities [2]{8}, resp.), and $\widetilde{\omega}_2=2m^2$ (with [1]{1}). One of the null eigenvectors in orbit space is a spurious projection mode, and the other one lies on the subspace tangent to ${\mathcal{Y}^0}_{1,1,1}$, on which $V_3(x)|_{c=0}$ is constant. Since there are no further eigenvalues, these extrema are minima independently of the values of a and b, as long as the potential remains bounded below. $V_3(\phi)|_{c=0}$ is invariant under SO(8) (because $\mathrm{Tr}\phi^4 = 1/2(\mathrm{Tr}\phi^2)^2$ in su(3)), spontaneously broken to SO(7). Thus, the 7 massless modes belonging to $\widetilde{\omega}_1$ are GBs arising from this spontaneous symmetry breaking. At the extrema $\pm {y^0}_{2,1}$ the SU(3) subgroup of stability is SU(2) × U(1), so there are 4 GBs, a massive U(1) mode and an SU(2) triplet of PGBs, which can acquire mass from SO(8) breaking interactions. This triplet contains the tangent mode. When $c\neq 0$ the spectrum at ${x^0}_{2,1,+}$ is as in (14), with $n=1$, and with ω_4 omitted. The triplet of PGBs acquires a mass of $\mathcal{O}(\sqrt{cm})$ from the explicit breaking of SO(8) due to $c\neq 0$.

Unlike the case $N=3$, for $N>3$ the symmetry SO(N^2-1) is explicitly broken by the dimension 4 operator $\mathrm{Tr}(\phi^4)$ in $V_N(\phi)$ so we expect it to play a limited role in the theory, and not to have any influence on the scalar mass spectrum for $|b|\gtrsim|a|$.

4 Renormalization group evolution

We consider now the RG evolution of the masses and the parameters in the potential. We consider the interaction of the scalar field ϕ with a fermion field in the fundamental representation of SU(N),

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu\phi\partial^\mu\phi\right) - V_N(\phi) + \overline{\psi} i\,\partial\!\!\!/\psi - \overline{\psi}(M+iM_5\gamma_5)\psi - \overline{\psi}\phi(g+ig_5\gamma_5)\psi\ . \tag{16}$$

The one-loop RG equations in MS scheme for the Yukawa couplings and fermion masses are,

$$\begin{aligned}
\mu\frac{dg}{d\mu} &\equiv \beta_g = \frac{1}{8\pi^2 N}(N^2-3)gg_5^2 + \frac{1}{8\pi^2 N}(N-1)(N+3)g^3 \\
\mu\frac{dg_5}{d\mu} &\equiv \beta_{g_5} = \frac{1}{8\pi^2 N}(N-1)(N+3)g_5 g^2 + \frac{1}{8\pi^2 N}(N^2-3)g_5^3
\end{aligned} \tag{17a}$$

$$\begin{aligned}
\mu\frac{dM}{d\mu} &= \frac{1}{8\pi^2 N}(N^2-1)\left(M(3g^2-g_5^2)+4gg_5M_5\right) \\
\mu\frac{dM_5}{d\mu} &= \frac{1}{8\pi^2 N}(N^2-1)\left(M_5(3g_5^2-g^2)+4gg_5M\right) \ .
\end{aligned} \tag{17b}$$

For the dimensionless parameters in V_N we have,

$$\mu\frac{da}{d\mu} \equiv \beta_a = \frac{2}{\pi^2}(N^2+7)a^2 + \frac{6}{\pi^2 N^2}(N^2+3)b^2 + \frac{4}{\pi^2 N}(2N^2-3)ab + \frac{1}{\pi^2}ag^2 \tag{18a}$$

$$\mu\frac{db}{d\mu} \equiv \beta_b = \frac{4}{\pi^2 N}(N^2-9)b^2 + \frac{24}{\pi^2}ab + \frac{1}{\pi^2}bg^2 - \frac{1}{8\pi^2}(g^2+g_5^2)^2 \ , \tag{18b}$$

and for the dimensionful ones,

$$\mu\frac{dc}{d\mu} = \frac{6}{\pi^2}c\left(\left(N-\frac{6}{N}\right)b+2a+\frac{1}{8}g^2\right) - \frac{1}{2\pi^2}(Mg+M_5g_5)(g^2+g_5^2) \tag{19a}$$

$$\begin{aligned}
\mu\frac{dm^2}{d\mu} = {}& \frac{2}{\pi^2}m^2\left((N^2+1)a+\frac{2N^2-3}{N}b+\frac{g^2}{4}\right) - \frac{9}{\pi^2 N}(N^2-4)c^2 \\
& + \frac{1}{\pi^2}(g^2+g_5^2)(M^2+M_5^2) + \frac{2}{\pi^2}(gM+g_5M_5)^2 \ .
\end{aligned} \tag{19b}$$

Some comments about these equations are in order. β_g and β_{g_5} have the same sign as g and g_5, resp., so $|g|$ and $|g_5|$ are monotonically increasing functions of μ, as expected. The coupling a is monotonically increasing (see below), but $|a|$ and b need not be and, therefore, the stability of the potential is not guaranteed, as discussed in detail below. The sign of the fermion contribution to β_b is independent of N, and of whether the scalar field is in the adjoint representation of SU(N) or U(N). In the case $N=1$ our sign agrees with textbook results (see ch. 5 of [3] and ch. 6 of [4]). If $M=0=M_5$, then $\mu dc/d\mu \propto c$, due to chiral symmetry. Eqs. (17a), (18) and (19b) agree with [5] when $g_5=0=M=M_5=c=0$.[2] An important verification is that for $N=3$ the quantities $\beta_a+1/2\beta_b$, $\mu dc/d\mu$ and $\mu dm^2/d\mu$ should not depend on a and b separately, but only on $a+b/2$. This test is passed by (18) and (19). [3]

The running Yukawa couplings are easily obtained by setting $g(t)=g_i f(t)$, $g_5(t)=g_{5i}f(t)$, $f(0)=1$, with g_i, g_{5i} the initial values, $f(t)$ an auxiliary function and $t\equiv\log(\mu/\mu_0)$, with $\mu\geqslant\mu_0>0$ and μ_0 a reference MS scale. Solving (17a) for $f(t)$ we get

$$g(t)=\frac{g_i}{\sqrt{1-t/t_{\rm f}}}\ , \quad g_5(t)=\frac{g_{5i}}{\sqrt{1-t/t_{\rm f}}}\ , \quad t_{\rm f}=\frac{4\pi^2N}{(N^2-3)g_{5i}^2+(N-1)(N+3)g_i^2}\ . \tag{20}$$

The fact that g and g_5 enter asymmetrically in β_g and β_{g5}, and therefore also in $t_{\rm f}$, is due to the scalar wave function renormalization not depending on g_5 at this order. From (20) we see that the ratio g/g_5 is RG invariant. With (20), (17b) can also be integrated

$$\begin{aligned}
M(t) &= \frac{g_i}{g_i^2+g_{5i}^2}(g_iM_i+g_{5i}M_{5i})\left(\frac{1}{1-t/t_{\rm f}}\right)^{\gamma_1} + \frac{g_{5i}}{g_i^2+g_{5i}^2}(g_{5i}M_i-g_iM_{5i})\left(\frac{1}{1-t/t_{\rm f}}\right)^{\gamma_2}\ , \\
M_5(t) &= \frac{g_{5i}}{g_i^2+g_{5i}^2}(g_iM_i+g_{5i}M_{5i})\left(\frac{1}{1-t/t_{\rm f}}\right)^{\gamma_1} - \frac{g_i}{g_i^2+g_{5i}^2}(g_{5i}M_i-g_iM_{5i})\left(\frac{1}{1-t/t_{\rm f}}\right)^{\gamma_2}\ , \\
\gamma_1 &= -3\gamma_2 = \frac{3}{8\pi^2}\frac{N^2-1}{N}t_{\rm f}(g_i^2+g_{5i}^2)\ .
\end{aligned} \tag{21}$$

[2] Except for the terms of $\mathcal{O}(g^4)$, which were considered of higher order in [5]. Here we take into account the full one-loop contribution.

[3] But failed by the RG eqs. in [2].

We remark that $\gamma_1 = 3/2 + \mathcal{O}(N^{-2})$ and $\gamma_2 = -1/2 + \mathcal{O}(N^{-2})$, independently of the asymptotic behavior of Yukawa couplings for large N. A combination with a particularly simple analytical expression is $g(t)M(t) + g_5(t)M_5(t) = (g_iM_i + g_{5i}M_{5i})(1/(1-t/t_{\rm f}))^{\gamma_1+1/2}$. Clearly, the one-loop approximation involved in (17) only holds for $|g|, |g_5| \ll 1$, so the exact solutions (20) and (21) are physically valid only for $t \ll t_{\rm f}$.

We consider now (18a). For a and g^2 fixed β_a is a quadratic polynomial in b and $\beta_a < 0$ is possible only if b lies between its roots. From the expression for those roots we can show that $\beta_a < 0$ implies that (2) does not hold. Thus, if V_N is bounded below then $\beta_a > 0$ and $a(t)$ is monotonically increasing. Like $g(t)$ and $g_5(t)$, $a(t)$ and $b(t)$ also have a mobile singularity at a finite t, whose location $t_{\rm s} > 0$ we can choose as a constant of integration. (There is a related singularity for $t < 0$ which will not be of interest to us). For $g^2 + g_5^2$ large enough the evolution of a and b is dominated by those couplings and $t_{\rm s} = t_{\rm f}$. We assume $g^2, g_5^2 \sim \mathcal{O}(|a|)$ or smaller. Under that assumption we have $t_{\rm s} < t_{\rm f}$ and, in fact, over large regions of parameter space $t_{\rm s} \ll t_{\rm f}$.

In the region near the singularity $b(t)$ can be neglected in (18a) and, with (20), it can be integrated to give

$$
\begin{aligned}
a(t) &\simeq \frac{\pi^2}{2(N^2+7)} \frac{A}{(t_{\rm f}-t)\left(1-\left(\frac{t_{\rm f}-t_{\rm s}}{t_{\rm f}-t}\right)^A\right)} \simeq \frac{\pi^2}{2(N^2+7)}\frac{1}{t_{\rm s}-t} \ , \\
A &= \frac{(N+1)(N-3)g_i^2 + (N^2-3)g_{5i}^2}{(N-1)(N+3)g_i^2 + (N^2-3)g_{5i}^2} \ .
\end{aligned}
\tag{22}
$$

The second equality on the first line holds for $t \ll t_{\rm f}$. If $g_i = 0$ we must set $t_{\rm f} = \infty$ in (22). For $|b_i| < a_i$ the expression (22) gives a good approximation to $a(t)$ also in the physically relevant region $t \ll t_{\rm s}$. Furthermore, for $N > 3$ we see that $A \simeq 1$ and from the initial condition $a(0) = a_i$ we obtain [4]

$$
t_{\rm s} \simeq \frac{\pi^2}{2(N^2+7)a_i} \ . \tag{23}
$$

If $|b_i| > |a_i|$ the approximate solution (22) does not hold. There is a critical value t_1, however, such that $a(t) > |b(t)|$ for all $t_1 < t < t_{\rm s}$. Thus, for $|b_i| > |a_i|$ we can find an approximate solution that interpolates between the regime $|b(t)| > |a(t)|$ $(0 \leq t < t_1)$ and the regime $|b(t)| < a(t)$ where (22) holds.

We are interested in the UV stability of the potential and of the vacuum. By UV stability of the potential we mean that (2) is satisfied for all $0 < t < t_{\rm s}$ if it is satisfied at $t = 0$. Similarly, we say that the vacuum is UV stable if it is a minimum of V_N for $0 \leq t < t_{\rm s}$. In the discussion of UV stability that follows the relevant regime is $|b(t)| < a(t)$ so we assume $|b_i| < a_i$ for that purpose (but not in the numerical solutions given below).

Stability of the vacuum requires $b > 0$ (see (15)). However, the condition $b_i > 0$ is not enough to guarantee that $b(t) > 0$ for all $0 < t < t_{\rm s}$, since the last term in (18b) can drive $b(t)$ to negative values. Once b becomes negative it diverges to $-\infty$ at $t = t_{\rm s}$ driven by the last three terms in (18b). There is a minimal value $b_{\rm min} > 0$ such that $b(t) > 0$ for all $0 < t < t_{\rm s}$ if and only if $b_i > b_{\rm min}$. In order to obtain an approximate expression for $b_{\rm min}$ the relevant regime is $b < a$, so we can neglect the first term on the rhs of (18b) and integrate the resulting linear eq. with $a(t)$ from (22) and g, g_5 from (20), to get,

$$
b(t) \simeq \left(\frac{1}{1-t/t_{\rm s}}\right)^{12/(N^2+7)} \left(e^{g_i^2 t/\pi^2}\left(b_i - \frac{1}{8}\frac{(g_i^2+g_{5i}^2)^2}{g_i^2}\right) + \frac{1}{8}\frac{(g_i^2+g_{5i}^2)^2}{g_i^2}\right) \ . \tag{24}
$$

This approximate solution, together with (23), leads to

$$
b_{\rm min} \simeq \frac{1}{8}\frac{(g_i^2+g_{5i}^2)^2}{g_i^2}\left(1 - e^{-g_i^2/(2(N^2+7)a_i)}\right) \ . \tag{25}
$$

The condition $b_i > b_{\rm min}$ is not very restrictive since, for $g_i^2, g_{5i}^2 \sim \mathcal{O}(a_i)$, typically $b_{\rm min} \ll a_i$. The accuracy of the approximate relation (25) is best illustrated by quoting some representative numerical values. For $a_i = 1/600$, $g_i = 1/10 = g_{5i}$, (25) gives a result for $b_{\rm min}$ that differs from the result obtained numerically

[4] From (20) and (23), $t_{\rm s} < t_{\rm f} \iff (N^2-3)g_{5i}^2 + (N-1)(N+3)g_i^2 < 8\pi^2 N(N^2+7)a_i$. Notice that the rhs is cubic in N and the lhs is quadratic. With $8\pi^2 \simeq 80$ and g_i^2, $g_{5i}^2 \simeq \mathcal{O}(a_i)$, the inequality is satisfied in large open regions of parameter space.

from (18b) by 35% for $N = 3$, 10% for $N = 6$ and 3% for $N = 12$. The accuracy is higher for larger values of N.

Since we require $b > 0$ for $0 \leq t < t_{\rm s}$, and since a is monotonically increasing, if $a_i > 0$ then (2) are trivially satisfied for all $0 \leq t < t_{\rm s}$ and V_N is UV stable. If $a_i < 0$, but such that (2) are satisfied at $t = 0$ with $b_i > b_{\rm min}$, (2) can in principle be violated at $t > 0$ if $b(t) > 0$ decreases rapidly enough. This imposes bounds on the possible values of $a_i < 0$, g_i and g_{5i}, which must be satisfied in order to ensure that $Na + b$ remains positive throughout the evolution. Since, however, $a(t)$ is monotonically increasing and $b(t) > 0$, (2) will be satisfied for all values of t ($< t_{\rm s}$) larger than a certain critical value. We will not dwell longer on this, and from now on we assume $a_i > 0$, $b_i > b_{\rm min} > 0$ so that $a(t), b(t) > 0$ for all $0 \leq t < t_{\rm s}$.

We assume V_N to be of the form (1) with $c \geq 0$, the case $c < 0$ can be obtained by means of the transformation $\phi \to -\phi$. The fermion masses M, M_5 and Yukawa couplings g, g_5 can be $\lesseqgtr 0$, their signs entering the scalar sector RG eqs. (18), (19) only through $Mg + M_5 g_5$ in (19a). With the assumptions $a_i > 0$, $b_i > b_{\rm min}$, and $c_i > 0$, if $Mg + M_5 g_5 < 0$ then $dc/dt > 0$ and $c(t)$ is monotonically increasing, therefore positive throughout the evolution. If $Mg + M_5 g_5 > 0$, however, even if $c_i > 0$ it may happen that $c(t)$ becomes negative, thus diverging to $-\infty$ at $t = t_{\rm s}$. The classical extremum ${x^0}_{2n,n,+}$ is a saddle-point for $c < 0$. As in the case of $b(t)$ discussed above, we will have $c(t) > 0$ for all $0 \leq t < t_{\rm s}$ only if $c_i > c_{\rm min}$ for some critical value $c_{\rm min}$. An approximate solution to (19a), valid for $b_i < a_i$, can be obtained by substituting (22), (24) (with $\exp(g_i^2 t/\pi^2) \simeq 1$), (20) and (21) into (19a) to obtain,

$$
\begin{gathered}
c(t) \simeq \left(\frac{1}{1 - t/t_{\rm s}}\right)^{6/(N^2+7)} \left(e^{\rho t}(c_i - \widetilde{c}_i) + \widetilde{c}_i\right) \ , \\
\rho = \frac{3}{4\pi^2} g_i^2 + \frac{6}{\pi^2}\frac{N^2-6}{N}\frac{N^2+7}{N^2-5} b_i \ , \quad \widetilde{c}_i = \frac{1}{2\pi^2 \rho}(M_i g_i + M_{5i} g_{5i})(g_i^2 + g_{5i}^2) \ .
\end{gathered}
\tag{26}
$$

From here, we get,

$$
c_{\rm min} \simeq \left(1 - e^{-\rho t_{\rm s}}\right) \widetilde{c}_i \ . \tag{27}
$$

Notice that for $Mg + M_5 g_5 < 0$ we have $c_i > 0 > c_{\rm min}$, so $c(t) > 0$ for all $0 < t < t_{\rm s}$. Substituting (23) in (27) we obtain an approximate expression for $c_{\rm min}$ solely in terms of initial values. To give an estimate of the accuracy of that approximation, for $a_i = 1/300$, $b_i = 1/450$, $g_i = -1/15 = g_{5i}$, $M_i = M_{5i} = m_i = 1$, the value of $c_{\rm min}$ obtained with (27) and (23) differs from the more accurate value computed numerically from (19a) by 35%, 23% and 11% for $N = 3$, 6 and 12, resp.

The evolution of $m^2(t)$ may be dominated by the term $\propto c^2$ in (19b), for $c^2(t)$ sufficiently large, causing $m^2(t)$ to decrease and eventually become negative.[5] An upper bound $c_{\rm max}$ on the initial value c_i exists, so that $c_i < c_{\rm max}$ ensures that $m^2(t) > 0$ for $0 \leq t < t_{\rm s}$. Using the exact solutions for the fermion parameters and the approximate ones for a, b, c given above, we find the following approximate form from $m^2(t)$, valid for $0 < b_i < a_i$,

$$
\begin{gathered}
m^2(t) \simeq \left(\frac{1}{1 - t/t_{\rm s}}\right)^{\frac{N^2+1}{N^2+7}} \left(m_i^2 e^{\eta t} + \widetilde{m}_i^2 t \left(1 - \frac{1}{2}\frac{t}{t_{\rm s}}\right)\right) \ , \qquad \eta = \frac{2}{\pi^2}\frac{2N^2-3}{N} b_i + \frac{g_i^2}{2\pi^2} \ . \\
\widetilde{m}_i^2 = \frac{1}{\pi^2}\left(-9\frac{N^2-4}{N} c_i^2 + (g_i^2 + g_{5i}^2)(M_i^2 + M_{5i}^2) + 2(g_i M_i + g_{5i} M_{5i})^2\right) \ .
\end{gathered}
\tag{28}
$$

From this approximate solution we obtain, using (23),

$$
c_{\rm max} \simeq \frac{1}{3}\sqrt{\frac{N}{(N^2-4)}} \left(4(N^2+7) a_i m_i^2 + (g_i^2 + g_{5i}^2)(M_i^2 + M_{5i}^2) + 2(g_i M_i + g_{5i} M_{5i})^2\right)^{1/2} \ . \tag{29}
$$

For the set of parameters quoted below (27) the values for $c_{\rm max}$ obtained from (29) differ from those obtained numerically from (19b) by 23%, 9% and 3% for $N = 3$, 6 and 12, resp.

[5] Typically, ω_4 changes sign at lower scales than m^2 does (see below), in principle precluding the possibility of radiative symmetry breaking, at least in this vacuum.

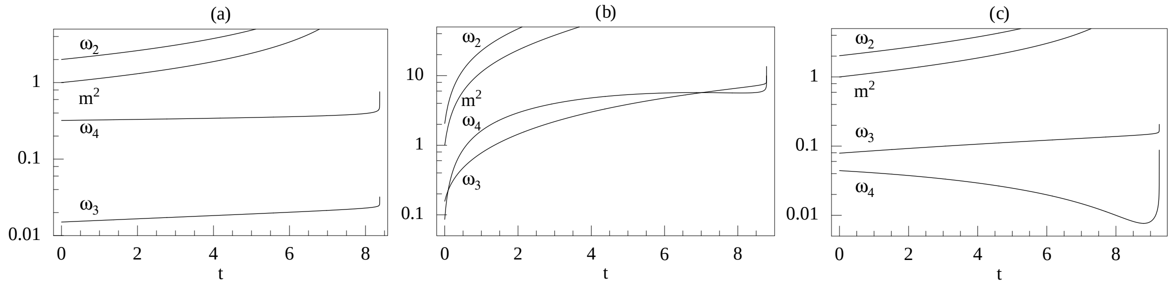

Figure 1: RG evolution of $m^2(t)/m_i^2$ and $\omega_{2,3,4}/m_i^2$, for $Mg + M_5 g_5 < 0$. In all cases $N = 12$, $a_i = 1/300$. (a) $b_i = 1/300$, $c_i = 1/1000$, $g_i = -1/20 = g_{5i}$, $M_i = 1 = M_{5i}$, (b) $b_i = 1/450$, $c_i = 1/100$, $g_i = -1/10 = g_{5i}$, $M_i = 20$, $M_{5i} = 35$, (c) $b_i = 1/900$, $c_i = 1/200$, $g_i = -1/10 = g_{5i}$, $M_i = 3/2 = M_{5i}$. The end values of a are: (a) $a(8) = 0.09$, (b) $a(8.5) = 0.11$, (c) $a(9) = 0.13$.

A further condition for the stability of the vacuum is $\omega_4 > 0$. From (15) we see that such condition can be formulated as $r(t) < 1$, with $r = \sqrt{N/2}a^{1/2}c/(bm)$. A necessary condition for $r(t) < 1$ throughout the evolution is obviously $r(0) < 1$,

$$\sqrt{\frac{N}{2}} a_i^{1/2} c_i < b_i m_i \ . \tag{30}$$

The dependence of $r(t)$ on t is not monotonic in general, so imposing (30) and $r(t_s{}^-) < 1$ is not sufficient to guarantee that $r(t) < 1$ for all $t < t_s$. Furthermore, $r(t)$ may have rapid variations, which makes it difficult to obtain approximate expressions for it that remain accurate over broad regions of initial values in parameter space. For these reasons, we will not attempt to give a single bound on initial conditions that ensures $\max_{0 \leq t < t_s}(r(t)) < 1$. Rather, we require initial conditions to obey (30), and study the evolution of ω_4 numerically. The stability condition $\omega_4 > 0$ implies an upper bound on c_i which is typically more restrictive than (29).

Fig. 1 shows the evolution of the mass-squared eigenvalues (14) for $Mg + M_5 g_5 < 0$. As shown there, the location t_s of the RG flow singularity has a small dependence on b_i, not taken into account in (23) which is strictly valid only for $b_i \ll a_i$. $\omega_{3,4}$ remain much smaller than ω_2 and m^2 throughout the evolution, even for $b \gtrsim a$, when the broken $SO(N^2 - 1)$ symmetry should be irrelevant.

Depending on the values of b_i and c_i, ω_3 can be smaller, larger, or approximately equal to ω_4 throughout the evolution, as seen in fig. 1 and (15). The effect of fermion masses on the evolution of scalar ones is apparent in fig. 1b. In that figure the value of c is close to the upper bound imposed by (30), which is reflected in the small value of ω_4 at the beginning of the evolution. Similarly, a slightly larger value of c in fig. 1c would cause the dip at the end of the evolution of ω_4 to reach negative values.

For $Mg + M_5 g_5 > 0$ the RG evolution of the mass-squared spectrum remains qualitatively the same as in fig. 1, except for the case $c \simeq c_{\min}$ in which $\omega_3(t)$ (as well as $c(t)$) is monotonically decreasing over essentially all of the interval $0 \leq t < t_s$. This decrease is shown in fig. 2, where we chose a small value for a_i and $b_i \simeq 10a_i$ in order to obtain a larger t_s and a small value of $c_{\min}$. For the parameters used in the figure, with $c_i \simeq c_{\min}$, ω_3 is separated from the rest of the spectrum by a factor 10^4–10^6 throughout the evolution.

5 Final remarks

At the beginning of sect. 2 we describe the qualitative motivations for this paper. The two possibilities mentioned there are realized in the model defined by (1) and (16), at the classical vacuum $x^{0}{}_{2n,n,+}$ defined in sect. 3. First, in the case $N = 3$ the scalar sector is invariant under SO(8), spontaneously broken to SO(7) and explicitly broken by the fermion sector and also, softly, by the cubic scalar self-coupling if $c \neq 0$. That explicit breaking gives mass to a triplet of PGBs, as described in sect. 4. When $c = 0$ there is a manifold of minima in orbit space, $\mathcal{Y}^{0}{}_{1,1,1}$ described by (7), which collapses to the two points $x^{0}{}_{2,1,\pm}$ for $c \neq 0$. Second, in the case $N = 3n > 3$ there is no larger symmetry group containing SU(N), and no explicit symmetry breaking. When $c = 0$ the manifold $\mathcal{Y}^{0}{}_{n,n,n}$ of (7) in orbit space comprises only saddle points, and collapses

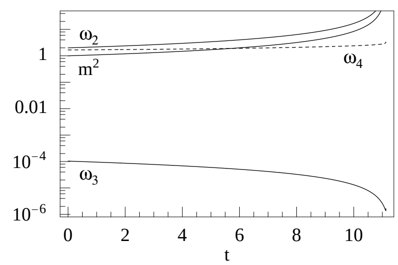

Figure 2: As in fig. 1, but for $Mg + M_5 g_5 > 0$. $N = 12$, $a_i = 1/1000$, $b_i = 1/100$, $c_i = 5.35 \times 10^{-6}$, $g_i = 1/30 = g_{5i}$, $M_i = 1/10 = M_{5i}$. The end value of a is $a(10.5) = 0.05$.

into ${x^0}_{2n,n,\pm}$ for $c > 0$. At the minimum ${x^0}_{2n,n,+}$ there is an SU($2n$) multiplet, containing the mode tangent to ${\mathcal{Y}^0}_{n,n,n}$ when $c \to 0$, with mass $\propto \sqrt{cm}$. This happens even if (at a given renormalization scale μ) we set $b \gg |a| \geq 0$ so that, unlike the case $N = 3$, there is no hint of SO($N^2 - 1$) symmetry in $\mathcal{L}$. We remark that, as long as $b \neq 0$, there is no eigenvalue proportional to c in the spectrum (11) at any extremum ${x^0}_{n_1,n_2,\pm}$ except for $n_1 = 2n_2$.

In section 4 we discuss in detail the RG running of couplings and masses. Approximate bounds are found on the initial values of b and c which are necessary for UV stability of the vacuum. Numerical study of the evolution of m^2 and $\omega_{2,3,4}$ (figs. 1 and 2) shows that ω_3 (and, for $b \ll a$, also ω_4) remains much smaller than m^2. For $gM + g_5 M_5 < 0$, $c(t)$ is monotonically increasing but for $gM + g_5 M_5 > 0$ we have $c > 0$ (as needed for $\omega_3 > 0$) only if $c(0) > c_{\min}$, with $c_{\min}$ approximately given (27). Interestingly, for $c(0)$ close enough to (but still larger than) its lower bound the mass $\sqrt{\omega_3}$ of the "tangent mode" multiplet is a monotonically decreasing function of the renormalization scale throughout the evolution.

For $N > 3$ the vacuum ${x^0}_{2n,n,+}$ is not a global minimum, which in the simple model discussed here is not important. In fact, even in realistic models a metastable vacuum is not in conflict with phenomenology if long–lived enough. Although the mass splittings we obtain with this model are somewhat modest, we may speculate that the mechanism described in this paper could lead to more elaborate theories in which the massless modes in the spectrum are removed by the Higgs mechanism and the heavier ones are pushed up by radiative corrections to very high mass scales, so that only the lighter massive modes (corresponding to ω_3 in fig. 2) remain in the physically accesible spectrum.

References

[1] L.-F. Li, Phys. Rev. **D 9** (1974), 1723.

[2] T. Murphy, L. O'Raifeartaigh, Nucl. Phys. **B 229** (1983), 509.

[3] S. Coleman, *Aspects of Symmetry*, Cambridge U. Press, Cambridge, 1993.

[4] T.-P. Cheng, L.-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford U. Press, Oxford, 1991.

[5] A. O. Bouzas, Int. J. Mod. **A 18** (2003), 3695; ibid. **21** (2006), 1157.

A Projected Hessian matrix for constrained extrema

Let $V : \mathbb{R}^N \to \mathbb{R}$ and $g_a : \mathbb{R}^N \to \mathbb{R}$, $1 \leq a \leq A < N$, be smooth functions. Let $\mathcal{G} \subset \mathbb{R}^N$ be the implicit manifold defined by the set of equations $g_a(x) = 0$. We assume that $\mathrm{rank}(\partial_j g_a(x)) = A$ for all $x \in \mathcal{G}$, so that $\mathcal{G}$ is a smooth manifold. We are interested in the extrema of $V|_{\mathcal{G}} : \mathcal{G} \to \mathbb{R}$. Let x^0 be such an extremum, $c : [-1, 1] \to \mathcal{G}$ a smooth curve such that $c(0) = x^0$, and $\widetilde{V}(t) = V(c(t))$. We must have $\dot{\widetilde{V}}(0) = \partial_i V(x^0) \dot{c}_i(0) = 0$, and since this must be true for any vector $\dot{c}(0)$ tangent to $\mathcal{G}$ at x^0, we conclude that $\partial_i V(x^0) = \sum_{a=1}^{A} \lambda_a^0 \partial_i g_a(x^0)$ for some set of numbers λ_a^0. Thus, we are led to an extremization problem for $\mathcal{L}(x,\lambda_a) = V(x) - \sum_{a=1}^{A} \lambda_a \partial_i g_a(x)$, which is the method of Lagrange multipliers.

The sign of $\ddot{\tilde{V}}(0) = \partial^2_{ij} V(x^0)\dot{c}_i(0)\dot{c}_j(0) + \partial_i V(x^0)\ddot{c}_i(0)$ determines the nature of the extremum x^0. Twice differentiating the relation $g(c(t)) = 0$ and expressing $\partial_i V(x^0)$ in terms of Lagrange multipliers we get

$$\ddot{\tilde{V}}(0) = \left(\partial^2_{ij} V(x^0) - \sum_{a=1}^{A} \lambda_a^0 \partial^2_{ij} g(x^0)\right) \dot{c}_i(0)\dot{c}_j(0) \ . \tag{31}$$

For x^0 to be a minimum (resp. maximum) the expression on the l.h.s. must be positive (negative) for all tangent vectors $\dot{c}(0)$. Therefore, the Hessian matrix in which we are interested is

$$\widetilde{H}_V(x^0) = P_{\mathcal{G}}(x^0) \left(H_V(x^0) - \sum_{a=1}^{A} \lambda_a^0 H_{g_a}(x^0)\right) P_{\mathcal{G}}(x^0) \ , \tag{32}$$

where $(H_V(x^0))_{ij} = \partial^2_{ij} V(x^0)$ and similarly $H_{g_a}(x^0)$, and $P_{\mathcal{G}}(x^0)$ is the projector onto the subspace tangent to $\mathcal{G}$ at x^0. In the simplest case in which the constraints g_a are mutually orthogonal ($\partial_k g_a \partial_k g_b \propto \delta_{ab}$) we have

$$(P_{\mathcal{G}})_{ij}(x^0) = \delta_{ij} - \sum_{a=1}^{A} \frac{1}{\partial_k g_a(x^0)\partial_k g_a(x^0)} \partial_i g_a(x^0) \partial_j g_a(x^0) \ . \tag{33}$$

Due to the projection $P_{\mathcal{G}}(x^0)$ the matrix $\widetilde{H}_V(x^0)$ has A spurious null eigenvectors that span the subspace normal to $\mathcal{G}$ at x^0 and can be taken to be $\partial g_a(x^0)$. The remaining $N - A$ genuine eigenvectors must be orthogonal to $\partial g_a(x^0)$, and therefore span the tangent subspace at x^0. Their eigenvalues define the nature of x^0 as an extremum of $V|_{\mathcal{G}}$.

The extremization problems considered in this paper involve a single linear constraint with $\partial g(x^0) = (1, 1, \ldots, 1)$. At a two-component extremum x^0 of the form (3c), such constraint leads to Hessian matrices of the form,

$$\widetilde{H}_V(x^0) = \left(\begin{array}{c|c} \omega_3 I_{n_1} + \kappa_1 G^{(n_1,n_1)} & \kappa_2 G^{(n_1,n_2)} \\ \hline \kappa_3 G^{(n_2,n_1)} & \omega_4 I_{n_2} + \kappa_4 G^{(n_2,n_2)} \end{array}\right) \ , \tag{34}$$

with $G^{(n_1,n_2)} \in \mathbb{R}^{n_1 \times n_2}$, $(G^{(n_1,n_2)})_{ij} = 1$, I_n the $n \times n$ identity matrix, and $\omega_{3,4}$, $\kappa_{1\cdots 4}$ some numerical coefficients. Such matrices are easily diagonalized. Let $\{v_1^{(n)}, \ldots, v_n^{(n)}\}$ be an orthogonal basis for $\mathbb{R}^n$ such that $v_1^{(n)} = (1, 1, \ldots, 1)$. We define the $n \times n$ orthogonal matrix $U^{(n)}$

$$U^{(n)}_{ij} = \frac{1}{\sqrt{v_j^{(n)} \cdot v_j^{(n)}}} (v_j^{(n)})_i \qquad \text{(no summation over } j) \ . \tag{35}$$

The matrix $G^{(n_1,n_2)} = v_1^{(n_1)} \otimes v_1^{(n_2)}$ then satisfies $\left(U^{(n_1)\dagger} G^{(n_1,n_2)} U^{(n_2)}\right)_{ij} = \sqrt{n_1 n_2}\,\delta_{i1}\delta_{j1}$. Let U be the $N \times N$ orthogonal matrix

$$U = \left(\begin{array}{c|c} U^{(n_1)} & 0 \\ \hline 0 & U^{(n_2)} \end{array}\right) \ . \tag{36}$$

With this definition the matrix $U^\dagger \widetilde{H}_V(x^0) U$ is block diagonal, with one block equal to $\omega_3 I_{n_1-1}$, another equal to $\omega_4 I_{n_2-1}$, and a 2×2 block $\left(\begin{smallmatrix} \omega_3 + \kappa_1 n_1 & \kappa_2 \sqrt{n_1 n_2} \\ \kappa_3 \sqrt{n_1 n_2} & \omega_4 + \kappa_4 n_2 \end{smallmatrix}\right)$ with eigenvalues $\omega_{1,2}$, one of which should vanish due to the projection. At an extremum of type (3a) the form of $\widetilde{H}_V(x^0)$ is analogous to (34), with 9 blocks instead of 4, and the diagonalization procedure is the same as above.

B Mass spectrum in su(N)

In the main text and in appendix A we considered the extrema of $V_N(x)$ and its Hessian matrix. From those we can immediately obtain the extrema and mass spectrum in orbit space. (The latter being the orbifold $\mathrm{su}(N)/\mathrm{SU}(N) = \mathbb{R}^N_c/\mathcal{P}_N$, with $\mathbb{R}^N_c = \{x \in \mathbb{R}^N / \sum_{j=1}^N x_j = 0\}$ and $\mathcal{P}_N$ the permutation group of N elements.) In this appendix we extend the Hessian matrix of $V_N(x)$ to $V_N(\phi)$ over $\mathrm{su}(N)$. In order to do so we expand

$$\phi = \sum_{i=1}^{N} \phi_i E^{(ii)} + \sum_{i<j=1}^{N} \rho_{ij} \frac{1}{\sqrt{2}} \left(E^{(ij)} + E^{(ji)} \right) + i \sum_{i<j=1}^{N} \eta_{ij} \frac{1}{\sqrt{2}} \left(E^{(ij)} - E^{(ji)} \right) \tag{37}$$

with $\left(E^{(ij)}\right)_{mn} = \delta_{im}\delta_{jn}$, $1 \leq i,j,m,n \leq N$, and use $\{\phi_i\}_{i=1}^N \cup \{\rho_{ij}\}_{i<j=1}^N \cup \{\eta_{ij}\}_{i<j=1}^N$ as our coordinates, constrained by $\sum_{i=1}^N \phi_i = 0$.[6] The Hessian matrix of $V_N(\phi)$ at a diagonal matrix $\varphi = \mathrm{diag}(x_1, \ldots, x_N)$ is easily computed if we disregard the tracelessness constraint,

$$\frac{\partial^2 V_N}{\partial\phi_l \partial\phi_k}(\varphi) = (H_V(x_1, \ldots, x_N))_{lk} \ , \tag{38a}$$

$$\frac{\partial^2 V_N}{\partial\rho_{ij} \partial\rho_{kl}}(\varphi) = \frac{\partial^2 V_N}{\partial\eta_{ij} \partial\eta_{kl}}(\varphi) \propto \delta_{ik}\delta_{jl} \ , \tag{38b}$$

$$\frac{\partial^2 V_N}{\partial\rho_{ij} \partial\phi_k}(\varphi) = \frac{\partial^2 V_N}{\partial\eta_{ij} \partial\phi_k}(\varphi) = \frac{\partial^2 V_N}{\partial\rho_{ij} \partial\eta_{kl}}(\varphi) = 0 \tag{38c}$$

where $H_V(x)$ is the Hessian matrix of $V_N(x)$. The effect of taking the constraint into account is to substitute $\widetilde{H}_V(x)$ for $H_V(x)$ in (38), as discussed in appendix A.

The matrix (38) is block diagonal. At an extremum of type (3c) the eigenvalues of the block $\widetilde{H}_V(x)$ are $\omega_{1,2,3,4}$ as given in (11). The other two diagonal blocks are already diagonal, as indicated in (38b). Explicit calculation shows that the diagonal entries in those blocks vanish if the index (k,l) is such that $x^0{}_k \neq x^0{}_l$ (see (3c)). If (k,l) is such that $x^0{}_k = x^0{}_l = \eta_1$ (resp. η_2) then $\partial^2 V/\partial\rho_{kl}\partial\rho_{kl} = \omega_3$ (resp. ω_4). Counting the number of such pairs of indices gives the multiplicity of $\omega_{3,4}$ in each of the blocks $\partial^2 V/\partial\rho_{kl}\partial\rho_{kl}$ and $\partial^2 V/\partial\eta_{kl}\partial\eta_{kl}$. Adding those multiplicities to the reduced ones in (11) results in the total multiplicities shown in that equation.

[6] Alternatively we could use $N-1$ unconstrained coordinates for the diagonal elements, by expanding in a basis with diagonal matrices $H^{(i)}$ satisfying $\mathrm{Tr}H^{(i)} = 0$, instead of $E^{(ii)}$, e.g., a Cartan–Weyl basis.